A New Method for Solving Steady Flow of a Third-Grade Fluid in a Porous Half Space Based on Radial Basis Functions

Saeed Kazem[a], Jamal Amani Rad[b], Kourosh Parand[b], and Saied Abbasbandy[a]

[a] Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran

[b] Department of Computer Sciences, Shahid Beheshti University, G.C., Tehran, Iran

Reprint requests to K. P.; E-mail: saeedkazem@gmail.com

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In this study, flow of a third-grade non-Newtonian fluid in a porous half space has been considered. This problem is a nonlinear two-point boundary value problem (BVP) on semi-infinite interval. We find the simple solutions by using collocation points over the almost whole domain $[0,\infty)$. Our method based on radial basis functions (RBFs) which are positive definite functions. We applied this method through the integration process on the infinity boundary value and simply satisfy this condition by Gaussian, inverse quadric, and secant hyperbolic RBFs. We compare the results with solution of other methods.

Key words: Third-Grade Fluid; Porous Half Space; Radial Basis Functions; Positive Definite RBFs; Collocation Method.

Mathematics Subject Classification 2000: 34B15, 34B40

1. Introduction

1.1. Introduction of the Problem

The non-Newtonian fluids have been studied extensively for the past few decades because of their relevance to many industrial and natural problems. Many materials such as polymer solutions or melts, drilling muds, clastomers, certain oils and greases, and many other emulsions are classified as non-Newtonian fluids. The fluids of the differential type have received special attention between the many models which have been used to describe the non-Newtonian behaviour exhibited by certain fluids. The fluids of second and third-grade have been studied successfully in various types of flow situations which form a subclass of the fluids of the differential type. The third-grade fluid models even for steady flow exhibit such characteristics. The present study deals with the problem of non-Newtonian fluid of third-grade in a porous half space. The viscoelastic flows in porous space are extremely current in many engineering fields such as enhanced oil recovery, paper and textile coating, and composite manufacturing processes. Also the modelling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry model, for example, capillary tubes, undulating tubes, packs of spheres or cylinders [1, 2].

1.2. Introduction of the Radial Basis Functions

Radial basis functions (RBFs) interpolation are techniques for representing a function starting with data on scattered nodes. This technique first appears in the literature as a method for scattered data interpolation, and the method was highly favoured after being reviewed by Franke [3], who found it to be the most impressive of the many methods he tested. Later, Kansa [4, 5] in 1990 proposed an approximate solution of linear and nonlinear differential equations (DEs) using RBFs. For the last years, the RBFs method was known as a powerful tool for the scattered data interpolation problem. The main advantage of numerical methods which use radial basis functions is the meshless characteristic of these methods. The use of radial basis functions as a meshless method for the numerical solution of ordinary differential equations (ODEs) and partial differential equations (PDEs) is based on the collocation method. Kansa's method has recently received a great deal of attention from researchers [6 – 11].

Recently, Kansa's method was extended to solve various ordinary and partial differential equations including the nonlinear Klein–Gordon equation [10], regularized long wave (RLW) equation [12], high-order ordinary differential equations [13], the case of heat transfer equations [14], Hirota–Satsuma coupled Kortheweg–de Vries (KdV) equations [15], second-order parabolic equation with nonlocal boundary conditions [16], second-order hyperbolic telegraph equation [17], and so on.

All of the radial basis functions have global support, and in fact many of them, such as multiquadrics (MQ), do not even have isolated zeros [10, 12, 18]. The RBFs can be compactly and globally supported, are infinitely differentiable, and contain a free parameter c, called the shape parameter [12, 18, 19]. For more basic details about compactly and globally supported RBFs and convergence rate of them, the interested reader can refer to the recent books and paper by Buhmann [18, 20] and Wendland [21].

Despite many studies done to find algorithms for selecting the optimum values of c [22 – 24], the optimal choice of shape parameter is an open problem which is still under intensive investigation. For example, Carlson and Foley [23] found that the shape parameter is problem dependent. They observed that for rapidly varying functions, a small value of c should be used, but a large value should be used if the function has a large curvature [23]. Tarwater [24] found that by increasing c, the root-mean-square (RMS) of error dropped to a minimum and then increased sharply afterwards. In general, as c increases, the system of equations to be solved becomes ill-conditioned. Rippa [25] showed, numerically, that the value of the optimal c (the value of c that minimizes the interpolation error) depends on the number and distribution of data points, on the data vector, and on the precision of the computation. Cheng et al. [22] showed that when c is very large then the RBFs system error is of exponential convergence. But there is a certain limit for the value c after which the solution breaks down. In general, as the value of the shape parameter c increases, the matrix of the system to be solved becomes highly ill-conditioned and hence the condition number can be used for determining the critical value of the shape parameter for an accurate solution [22]. Recently, Roque and Ferreira [26] proposed a statistical technique to choose the shape parameter in radial basis functions. They use a cross-validation technique suggested by Rippa [25] for interpolation problems to find a cost function $\mathrm{Cost}(c)$ that ideally has the same behaviour as an error function. For some new work on optimal choice of shape parameter, we refer the interested reader to the recent work ofRoque and Ferreira [26] and Fasshauer and Zhang [27].

There are two basic approaches for obtaining basis functions from RBFs, namely direct approach (DRBF) based on a differential process [5] and indirect approach (IRBF) based on an integration process [8, 13, 28]. Both approaches were tested on the solution of second-order DEs and the indirect approach was found to be superior to the direct approach [8].

In contrast, the integration process is much less sensitive to noise [13, 29]. Based on this observation, it is expected that through the integration process, the approximating functions will be much smoother and therefore have higher approximation power [13, 29].

To numerically explore the IRBF methods with shape parameters for which the interpolation matrix is too poorly conditioned to use standard methods, the researchers used the contour-Padé (CP) algorithm [30, 31]. This is perhaps the major advantage of the IRBFs as RBFs methods are typically not employed in applications using the optimal shape parameters, but using some value of the parameter safely away from the region of ill-conditioning [31].

Some of the infinitely smooth RBFs choices are listed in Table 1. The RBFs can be of various types, for example: inverse quadrics (IQ), Gaussian forms (GA), hyperbolic secant (sech) form etc. Regarding the inverse quadratic, hyperbolic secant (sech), and Gaussian (GA), the coefficient matrix interpolating the RBFs is positive definite [32].

In this paper we apply the new method based on RBFs for solving the steady flow of a third-grade fluid in a porous half space. For convenience of the solution and to satisfy the infinity condition ($f(z) \to 0$ as $z \to \infty$), we use the three positive definite RBFs given in Table 1: 1 – Gaussian (GA) 2 – Inverse quadric (IQ), 3 – Secant hyperbolic (sech).

This paper is arranged as follows: in Section 2, we present a brief formulation ofthe problem. In Section 3,

Table 1. Some positive definite RBFs ($r = \|x - x_i\|$), $c > 0$.

Name of functions	Definition
Gaussian (GA)	$\frac{2}{\sqrt{\pi}} \exp(-cr^2)$
Inverse quadrics (IQ)	$1/(r^2 + c^2)$
Secant hyperbolic	$\mathrm{sech}(cr)$

we describe the properties of the radial basis functions. In Section 4 we implement the problem with the radial basis functions method, report our numerical finding, and demonstrate the accuracy of the proposed method. The conclusions are discussed in the final Section 5.

2. Problem Statement

In this section we focus on Hayat et al. [1] who have discussed the flow of a third-grade fluid in a porous half space. For unidirectional flow, they have generalized the relation [1]

$$(\nabla p)_x = -\frac{\mu\varphi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)u, \tag{1}$$

for a second-grade fluid to the following modified Darcy's Law for a third-grade fluid:

$$(\nabla p)_x = -\frac{\varphi}{k}\left[\mu u+\alpha_1\frac{\partial u}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2 u\right], \tag{2}$$

where u denote the fluid velocity, μ is the dynamic viscosity, and p is the pressure, k and φ, respectively represent the permeability and porosity of the porous half space which occupies the region $y > 0$, and α_1, β_3 are material constants. Defining the nondimensional fluid velocity f and the coordinate z:

$$\begin{aligned} &z=\frac{V_0}{\nu}y, \quad f(z)=\frac{u}{V_0}, \\ &V_0=u(0), \quad \nu=\frac{\mu}{\rho}, \end{aligned} \tag{3}$$

where ν and V_0 represent the kinematic viscosities. Then the boundary value problem modelling the steady state flow of a third-grade fluid in a porous half space becomes [1]

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+b_1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}-b_2 f\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2-b_3 f=0, \tag{4}$$

$$f(0)=1, \quad f(z)\to 0 \ \text{as} \ z\to\infty. \tag{5}$$

Where b_1, b_2, and b_3 are defined as

$$\begin{aligned} b_1 &= \frac{6\beta_3 {V_0}^4}{\mu\nu^2}, \\ b_2 &= \frac{2\beta_3\varphi {V_0}^2}{k\mu}, \\ b_3 &= \frac{\varphi\nu^2}{k{V_0}^2}. \end{aligned} \tag{6}$$

Above parameters are depended:

$$b_2 = \frac{b_1 b_3}{3}. \tag{7}$$

In [1], (4) is solved by a well-known analytical method, the homotopy analysis method (HAM). Recently, Ahmad [33] used an alternative approach to find an analytical solution of the problem. He gave the asymptotic form of the solution and utilized this information to develop a series solution.

3. Properties of Radial Basis Functions

3.1. Definition of the RBFs

Let $\mathbb{R}^+ = \{x\in\mathbb{R}, x\geq 0\}$ be the non-negative half-line and let $\phi:\mathbb{R}^+\to\mathbb{R}$ be a continuous function with $\phi(0)\geq 0$. A radial basis functions on $\mathbb{R}^d$ is a function of the form

$$\phi(\|X-X_i\|),$$

where X, $X_i\in\mathbb{R}^d$, and $\|.\|$ denotes the Euclidean distance between X and X_i. If one chooses N points $\{X_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$s(X)=\sum_{i=1}^N \lambda_i\phi(\|X-X_i\|), \quad \lambda_i\in\mathbb{R},$$

is called a radial basis functions as well [34].

The standard radial basis functions are categorized into two major classes [15]:

Class 1. Infinitely smooth RBFs [15, 35]:
These basis functions are infinitely differentiable and heavily depend on the shape parameter c, e.g. Hardy multiquadric (MQ), Gaussian (GA), inverse multiquadric (IMQ), and inverse quadric (IQ) (see Tab. 1).

Class 2. Infinitely smooth (except at centers) RBFs [15, 35]:
The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in Class 1. For example, thin plate spline, etc. [15].

3.2. RBFs Interpolation

The one dimensional function $y(x)$ to be interpolated or approximated can be represented by RBFs as

$$y(x) \approx y_N(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \boldsymbol{\Phi}^{\mathrm{T}}(x)\boldsymbol{\Lambda}, \tag{8}$$

where

$$\begin{aligned} \phi_i(x) &= \varphi(\|x - x_i\|), \\ \boldsymbol{\Phi}^{\mathrm{T}}(x) &= [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)], \\ \boldsymbol{\Lambda} &= [\lambda_1, \lambda_2, \ldots, \lambda_N]^{\mathrm{T}}, \end{aligned} \tag{9}$$

x is the input and $\{\lambda_i\}_{i=1}^N$ are the set of coefficients to be determined. By choosing N interpolate nodes $\{x_i\}_{i=1}^N$, we can approximate the function $y(x)$

$$y_j = \sum_{i=1}^{N} \lambda_i \phi_i(x_j), \quad j = 1, 2, \ldots, N.$$

To summarize discussion on coefficient matrix, we define

$$\boldsymbol{A}\boldsymbol{\Lambda} = \boldsymbol{Y},$$

where

$$\begin{aligned} \boldsymbol{Y} &= [y_1, y_2, \ldots, y_N]^{\mathrm{T}}, \\ \boldsymbol{A} &= [\boldsymbol{\Phi}^{\mathrm{T}}(x_1), \boldsymbol{\Phi}^{\mathrm{T}}(x_2), \ldots, \boldsymbol{\Phi}^{\mathrm{T}}(x_N)]^{\mathrm{T}}, \\ &= \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N) \end{bmatrix}. \end{aligned} \tag{10}$$

Note that $\phi_i(x_j) = \varphi(\|x_i - x_j\|)$ therefore we have $\phi_i(x_j) = \phi_j(x_i)$ and consequently $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$.

All the infinitely smooth RBFs choices listed in Table 1 will give the coefficient matrices $\boldsymbol{A}$ in (10) which are symmetric and nonsingular [32], i.e. there is a unique interpolant of the form (8), no matter how the distinct data points are scattered in any number of space dimensions. In the cases of inverse quadratic, inverse multiquadric (IMQ), hyperbolic secant (sech), and Gaussian (GA) the matrix $\boldsymbol{A}$ is positive definite and, for multiquadric (MQ), it has one positive eigenvalue and the remaining ones are all negative [32].

We have the following theorem about the convergence of RBFs interpolation:

Theorem: Assume $x_i, (i = 1, 2, \ldots, N)$, are N nodes in convex Ω, let

$$h = \max_{x \in \Omega} \min_{1 \le i \le N} \|x - x_i\|_2,$$

when $\hat{\phi}(\eta) < c(1 + |\eta|)^{-(2l+d)}$ for any $u(x)$ satisfies $\int (\hat{u}(\eta))^2 / \hat{\phi}(\eta) d\eta < \infty$, we have

$$\|u^{N(\alpha)} - u^{(\alpha)}\|_\infty \le c h^{l-\alpha},$$

where $\phi(x)$ is an RBF and the constant c depends on the RBFs, d is the space dimension, l and α are nonnegative integer. It can be seen that not only the RBF itself but also its any order derivative has a good convergence.

Proof. A complete proof is given by Wu [36, 37].

4. Solving the Model

In this problem, we use gaussian, inverse quadric, and secant hyperbolic RBFs (Table 1) which are positive definite functions and can get high accurate solutions [32]. Also these functions satisfy the infinity condition in (5).

Now we approximate $f'(z)$ and $f''(z)$ as

$$f'(z) \simeq f'_N(z) = \sum_{i=0}^{N} \lambda_i \phi_i(z), \tag{11}$$

$$f''(z) \simeq f''_N(z) = \sum_{i=0}^{N} \lambda_i \phi'_i(z). \tag{12}$$

By using integral operation $f(z)$ is obtained as

$$\begin{aligned} \int_z^\infty f'_N(t)\,\mathrm{d}t &= \sum_{i=0}^{N} \lambda_i \int_z^\infty \phi_i(t)\ \mathrm{d}t, \\ f_N(\infty) - f_N(z) &= \sum_{i=0}^{N} \lambda_i \int_z^\infty \phi_i(t)\ \mathrm{d}t, \\ f(z) \simeq f_N(z) &= \sum_{i=0}^{N} \lambda_i \int_\infty^z \phi_i(t)\ \mathrm{d}t. \end{aligned} \tag{13}$$

Equation (13) in the case of gaussian RBF gives

$$f_N(z) = \frac{1}{c} \sum_{i=0}^{N} \lambda_i (\mathrm{erf}(c(z - z_i)) - 1).$$

Also, $f_N(z)$ for cases of IQ-RBF and sech-RBF is obtained of the form

$$f_N(z) = \frac{1}{c}\sum_{i=0}^{N}\lambda_i\left(\frac{\pi}{2} - \arctan\left(\frac{z-z_i}{c}\right)\right), \quad \text{IQ-RBF},$$
$$f_N(z) = \frac{1}{c}\sum_{i=0}^{N}\lambda_i\left(\frac{\pi}{2} + \arctan\left(\sinh(cz-cz_i)\right)\right), \quad \text{sech-RBF}.$$

By substituting (11), (12), and (13) in (4), we define residual function

$$\text{Res}(z) = \frac{\mathrm{d}^2 f_N}{\mathrm{d}z^2} + b_1\left(\frac{\mathrm{d}f_N}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^2 f_N}{\mathrm{d}z^2} - b_2 f_N\left(\frac{\mathrm{d}f_N}{\mathrm{d}z}\right)^2 - b_3 f_N. \tag{14}$$

Now, by using N interpolate nodes $\{z_j\}_{j=0}^{N-1}$ plus a condition (5) we can solve the set of equations and consequently, the coefficients $\{\lambda_i\}_{i=0}^{N}$ will be obtained:

$$\begin{cases}\text{Res}(z_j) = 0, \quad j = 0,1,\ldots,N-1,\\ f_N(0) = 1.\end{cases} \tag{15}$$

Collocation points are chosen on an uniform grid $[0, z_\infty]$. Here, we choose $z_\infty = 30$ which satisfies $f(z_\infty) < \varepsilon$ with ε as a small positive value.

It is worth to mention that it is in general difficult to solve the nonlinear system (15) even by Newton's method. The main difficulty with such a system is how to choose the initial guess to handle Newton's method, in other words: How many solutions admit the system of nonlinear equations? We think the best way to discover a proper initial guess (or initial guesses) is to solve the system analytically for very small N (by using symbolic softwares program such as Mathematica or Maple) and then work out proper initial guesses and particularly multiplicity of solutions of such system. This action has been done by starting from proper initial guesses with a number of maximum iterations of ten.

5. Concluding Remarks

The non-Newtonian fluids have been studied extensively for the past few decades because of their relevance to many industrial and natural problems. In this paper we have shown the approximate solutions of flow of a third-grade fluid in a porous half space by three positive definite RBFs for some typical values of parameters, $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$. Here the numerical solution of $f'(0)$ is important. Ahmad [33] obtained this value by the shooting method and founded, correct to six decimal positions, $f'(0) = -0.678301$.

We compared the present method by using GA, IQ, and sech RBFs with the numerical solution and the Ahmad solution [33] in Tables 2, 3, and 4. The solutions are presented graphically in Figure 1.

The radial basis functions listed in Table 1 contain a shape parameter c that must be specified by the user. But here, by the meaning of residual function, we try to minimize $\|\text{Res}(z)\|^2$ by choosing a good shape parameter c [38]. We define $\|\text{Res}(z)\|^2$ as

$$\|\text{Res}(z)\|^2 = \int_0^b \text{Res}^2(z)\,\mathrm{d}z \simeq \sum_{j=0}^{m}\omega_j \text{Res}^2\left(\frac{b}{2}s_j + \frac{b}{2}\right),$$

where

$$\omega_j = \frac{b}{(1-s_j{}^2)\left(\frac{\mathrm{d}}{\mathrm{d}s}P_{m+1}(s)|_{s=s_j}\right)^2}, \quad j = 0,1,\ldots,m,$$
$$P_{m+1}(s_j) = 0, \quad j = 0,1,\ldots,m,$$

Table 2. Comparison between gaussian RBF solution and Ahmad solution [33] for $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ with $N = 20$ and $c = 0.1582$.

z	GA-RBF	Ahmad method [33]	Numerical [33]
0.0	1.00000000	1.00000	1.00000
0.2	0.87265264	0.87220	0.87260
0.4	0.76074843	0.76010	0.76060
0.6	0.66261488	0.66190	0.66240
0.8	0.57671495	0.57600	0.57650
1.0	0.50164542	0.50100	0.50140
1.2	0.43613322	0.43560	0.43590
1.6	0.32930679	0.32890	0.32920
2.0	0.24842702	0.24820	0.24840
2.5	0.17456033	0.17440	0.17450
2.7	0.15156849	0.15140	0.15160
3.0	0.12262652	0.12250	0.12260
3.4	0.09243890	0.09234	0.09242
3.6	0.08025570	0.08016	0.08024
4.0	0.06049038	0.06042	0.06047
4.2	0.05251411	0.05245	0.05250
4.4	0.04558868	0.04553	0.04558
4.6	0.03957597	0.03953	0.03957
4.8	0.03435598	0.03432	0.03435
5.0	0.02982446	0.02979	0.02982
$f'(0)$	−0.678301314	−0.681835	−0.678301
$\|\text{Res}\|^2$	$1.8554 \cdot 10^{-6}$	–	–

Table 3. Comparison between inverse quadric RBF solution and Ahmad solution [33] for $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ with $N = 20$ and $c = 17.04$.

z	IQ-RBF	Ahmad method [33]	Numerical [33]
0.0	1.00000000	1.00000	1.00000
0.2	0.87266001	0.87220	0.87260
0.4	0.76076870	0.76010	0.76060
0.6	0.66264542	0.66190	0.66240
0.8	0.57675020	0.57600	0.57650
1.0	0.50167989	0.50100	0.50140
1.2	0.43616288	0.43560	0.43590
1.6	0.32932216	0.32890	0.32920
2.0	0.24843091	0.24820	0.24840
2.5	0.17455938	0.17440	0.17450
2.7	0.15156786	0.15140	0.15160
3.0	0.12262741	0.12250	0.12260
3.4	0.09244181	0.09234	0.09242
3.6	0.08025912	0.08016	0.08024
4.0	0.06049365	0.06042	0.06047
4.2	0.05251686	0.05245	0.05250
4.4	0.04559078	0.04553	0.04558
4.6	0.03957740	0.03953	0.03957
4.8	0.03435682	0.03432	0.03435
5.0	0.02982483	0.02979	0.02982
$f'(0)$	−0.678301390	−0.681835	−0.678301
$\|\text{Res}\|^2$	$2.5364 \cdot 10^{-6}$	–	–

Table 4. Comparison between secant hyperbolic RBF solution and Ahmad solution [33] for $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ with $N = 20$ and $c = 0.0905$.

z	sech-RBF	Ahmad method [33]	Numerical [33]
0.0	1.00000000	1.00000	1.00000
0.2	0.87266081	0.87220	0.87260
0.4	0.76077103	0.76010	0.76060
0.6	0.66264901	0.66190	0.66240
0.8	0.57675439	0.57600	0.57650
1.0	0.50168395	0.50100	0.50140
1.2	0.43616626	0.43560	0.43590
1.6	0.32932345	0.32890	0.32920
2.0	0.24843050	0.24820	0.24840
2.5	0.17455844	0.17440	0.17450
2.7	0.15156713	0.15140	0.15160
3.0	0.12262720	0.12250	0.12260
3.4	0.09244225	0.09234	0.09242
3.6	0.08025974	0.08016	0.08024
4.0	0.06049429	0.06042	0.06047
4.2	0.05251737	0.05245	0.05250
4.4	0.04559109	0.04553	0.04558
4.6	0.03957750	0.03953	0.03957
4.8	0.03435674	0.03432	0.03435
5.0	0.02982462	0.02979	0.02982
$f'(0)$	−0.678301748	−0.681835	−0.678301
$\|\text{Res}\|^2$	$2.6279 \cdot 10^{-6}$	–	–

$P_{m+1}(s)$ is the $(m+1)$th-order Legendre polynomial and b is the biggest collocation node. Tables 2, 3, and 4 show the minimum of $\|\text{Res}(z)\|^2$ which is obtained with shape parameter c for cases of GA-RBF, IQ-RBF, and sech-RBF. The logarithmic graphs of the $\|\text{Res}(z)\|^2$ for GA, IQ, and sech RBFs at $b_1 =$

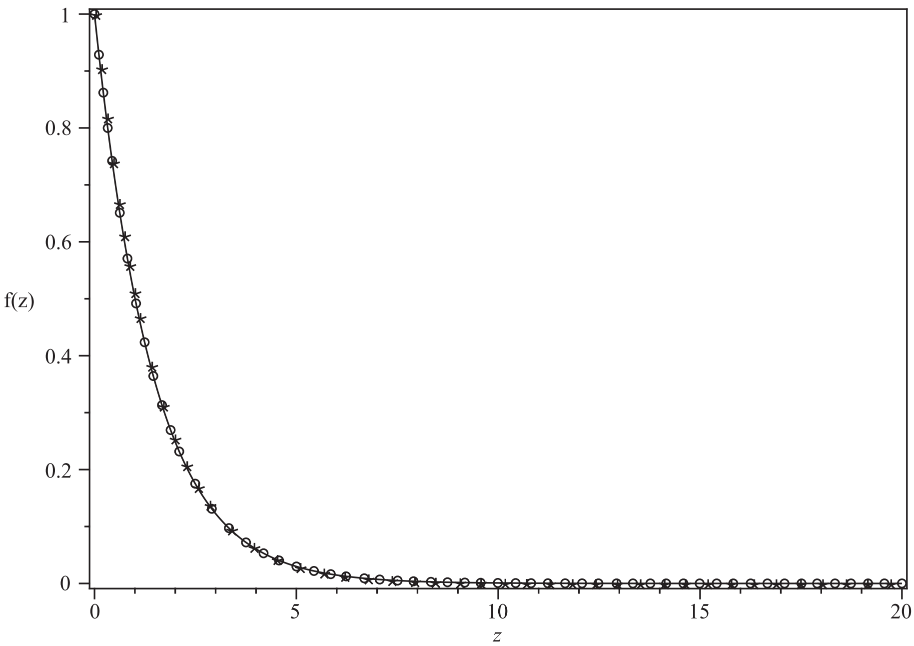

Fig. 1. Graphs of numerical approximate $f_N(z)$ by using GA-RBF ($\star$), IQ-RBF ($\circ$) and Sech-RBF (–).

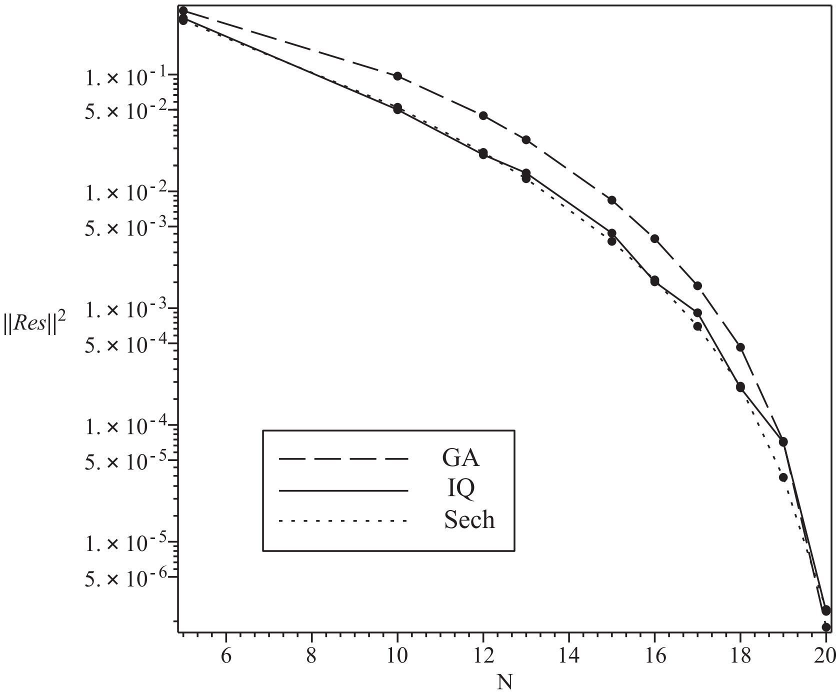

Fig. 2. Graphs of $||\mathrm{Res}||^2$.

0.6, $b_2 = 0.1$, and $b_3 = 0.5$ are shown in Figure 2. These graphs illustrate the convergence rate of the method. We find the simple solutions by using collocation points over almost the whole domain $[0,\infty)$. We applied this method through the integration process on the infinity boundary value and satisfy this condition.

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[1] T. Hayat, F. Shahzad, and M. Ayub, Appl. Math. Model. **31**, 2424 (2007).
[2] T. Hayat, F. Shahzad, M. Ayub, and S. Asghar, Commun. Nonlin. Sci. Numer. Simul. **13**, 1801 (2008).
[3] R. Franke, Math. Comput. **38**, 181 (1982).
[4] E. J. Kansa, Comput. Math. Appl. **19**, 127 (1990).
[5] E. J. Kansa, Comput. Math. Appl. **19**, 147 (1990).
[6] M. Sharan, E. J. Kansa, and S. Gupta, Appl. Math. Comput. **84**, 275 (1997).
[7] M. Zerroukat, H. Power, and C. Chen, Int. J. Numer. Meth. Eng. **42**, 1263 (1998).
[8] N. Mai-Duy and T. Tran-Cong, Neural Networks **14**, 185 (2001).
[9] M. Tatari, M. Dehghan, Eng. Anal. Bound. Elem. **34**, 206 (2010).
[10] M. Dehghan and A. Shokri, J. Comput. Appl. Math. **230**, 400 (2009).
[11] A. Alipanah and M. Dehghan, Appl. Math. Comput. **190**, 1754 (2007).
[12] S. U. Islam, S. Haqb, and A. Ali, J. Comput. Appl. Math. **223**, 997 (2009).
[13] N. Mai-Duy, Int. J. Numer. Meth. Eng. **62**, 824 (2005).
[14] K. Parand, S. Abbasbandy, S. Kazem, and A. Rezaei, Commun. Nonlin. Sci. Numer. Simul. **16**, 1396 (2011).
[15] A. J. Khattak, S. I. A. Tirmizi, and S. U. Islam, Eng. Anal. Bound. Elem. **33**, 661 (2009).
[16] M. Dehghan and M. Tatari, Numer. Meth. Part. D. E. **24**, 924 (2008).
[17] M. Dehghan and A. Ghesmati, Eng. Anal. Bound. Elem. **34**, 51 (2010).
[18] M. D. Buhmann, Acta Numer. **9**, 1 (2000).
[19] S. Sarra, Appl. Numer. Math. **54**, 79 (2005).

[20] M. D. Buhmann, Radial Basis Functions: Theory and Implementations, Cambridge University Press, New York 2004.
[21] H. Wendland, Scattered Data Approximation, Cambridge University Press, New York 2005.
[22] A. H. D. Cheng, M. A. Golberg, E. J. Kansa, and Q. Zammito, Numer. Meth. Part. D. E. **19**, 571 (2003).
[23] R. E. Carlson and T. A. Foley, Comput. Math. Appl. **21**, 29 (1991).
[24] A. E. Tarwater, A parameter study of Hardy's multiquadric method for scattered data interpolation, Report UCRL-53670, Lawrence Livermore National Laboratory, 1985.
[25] S. Rippa, Adv. Comp. Math. **11**, 193 (1999).
[26] C. Roque and A. Ferreira, Numer. Meth. Part. D. E. **26**, 675 (2010).
[27] G. Fasshauer and J. Zhang, Numer. Algorithms **45**, 346 (2007).
[28] N. Mai-Duy and T. Tran-Cong, Int. J. Numer. Meth. Fl. **37**, 65 (2001).
[29] W. R. Madych, S. A. Nelson, Approx. Theory Appl. **4**, 77 (1988).
[30] B. Fornberg and G. Wright, Comput. Math. Appl. **48**, 853 (2004).
[31] S. A. Sarra, Comput. Math. Appl. **51**, 1283 (2006).
[32] M. Powell, The Theory of Radial Basis Function Approximation in 1990, Clarendon, Oxford, 1992.
[33] F. Ahmad, Commun. Nonlin. Sci. Numer. Simul. **14**, 2848 (2009).
[34] M. Golberg, Eng. Anal. Bound. Elem. **23**, 285 (1999).
[35] M. Dehghan and A. Shokri, Numer. Algorithms **52**, 461 (2009).
[36] Z. Wu, J. Eng. Math. **19**, 1 (2002).
[37] Z. Wu and R. Schaback, IMA J. Numer. Anal. **13**, 13 (1993).
[38] K. Parand, S. Abbasbandy, S. Kazem, and J. A. Rad, Commun. Nonlin. Sci. Numer. Simul. **16**, 4250, 2011.